On the boundedness of fibered K-trivial varieties

(F-Hacon-Svaldi, Engel-F-Greer-Mauri-Svaldi)

$/\mathbb{C}$, projective varieties

Given X, the MMP predicts $X \overset{bir}{\dashrightarrow} X'$, mildly sing

$$\begin{array}{c} X' \\ \downarrow f \\ Z \end{array}$$

s.t.
1. $Z = pt$, $K_{X'}$ ample
2. $K_{X_z} \sim_{\mathbb{Q}} 0$ for $z \in Z$ general
3. $-K_{X_z}$ ample, $z \in Z$ general

} repeat for Z

it is known up to dim 3 & many special cases

$\Rightarrow$ there are 3 fundamental building blocks X

ⓐ K_X ample: canonical model

ⓑ $K_X \sim_{\mathbb{Q}} 0$, K-torsion variety

ⓒ $-K_X$ ample: Fano variety

Goal: study each building block and construct moduli spaces for them

Preliminary question: when does a class of varieties admit a finite dimensional parameter space?

Def.: A class of varieties $\{X_i\}_{i\in I}$ is called (birationally) bounded if there is a projective morphism of schemes of finite type $\mathcal{X} \to T$ s.t., $\forall X_i$, $\exists t \in T$ closed s.t. $X_i \cong \mathcal{X}_t$ $(X_i \underset{bir}{\simeq} \mathcal{X}_t)$

Example: • hypersurfaces of a given degree are bounded

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))) & \supseteq & \mathcal{X} = \{(x,f) \mid f(x)=0\} \\ \downarrow & & \downarrow \\ \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))) & = & T \end{array}$$

• $\{\mathbb{F}_n\}_{n\in\mathbb{N}}$ are not bounded, yet birationally bounded

$$\begin{array}{ccc} \mathbb{F}_n \overset{bir}{\dashrightarrow} \mathbb{P}^2 & = & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & = & T \end{array}$$

How about the building blocks?

- canonical models (mild sing) in $\dim = n$ are bounded if we fix the invariant K_X^n:
  - C, $g(C)=g\geq 2$ Riemann
  - $\dim = 2$ Bombieri
  - HMX

- Fano varieties in $\dim = n$ are bounded if we fix the singularities:
  - $\dim = 2$ Fano
  - smooth KMM
  - singular Birkar

- K-torsion varieties: • $\dim = 1$: elliptic curves are bdd
  - $\dim = 2$: K3s, ASs are not bdd, but are analytically bounded
    Enriques, bielliptic are bdd
  - $\dim \geq 3$: ???

Beyond the MMP, there are further decomposition theorems for K-torsion varieties

Thm. (BB decomposition): A K-torsion variety X with klt singularities admits a quasi-étale cover $X' \to X$ s.t.

mild: e.g., quotient

$$X' = A \times \left(\prod Y_i\right) \times \left(\prod Z_j\right)$$

- A is an ab. var
- Y_i is irreducible CY (ICY)
- Z_j is irreducible symplectic (IS)

we admit Spec $\mathbb{C}$ if a factor does not occur

if smooth: ICY $\Leftrightarrow \pi_1(Y) = 1$, $h^i(Y, \mathcal{O}_Y) = 0$ $0 < i < \dim Y$

for simplicity

IPS $\Leftrightarrow \pi_1(Z) = 1$ $\bigoplus H^0(Z, \Omega_Z^{u})$ generated by a symplectic 2-form

We can study boldness of these special types of K-torsion varieties

Current expectations: in each dimension
- AV are analytically bdd ✓
- IS are analytically bdd — all dim ≤ 2
- ICY are algebraically bdd — ???

Note: ICY make sense from dim ≥ 3 and given $h^2(X, \mathcal{O}) = 0$, all small deformations are algebraic

Only major evidence up until recent works

<u>Thm. (Gross):</u> Elliptic ICY 3-folds $X \to S$ are birationally bounded

Relevance:
- only result
- relevant for string theory
- database of $\approx 100'000$ predicts $b_2(X) \gg 0 \Rightarrow$ ell fibered

<u>Plan:</u> study fibered ICY & IPS
- more tools
- hope ICY with $b_2 \gg 0$ admit fibrations
- SYZ: expect IPS with $b_2 \geq 5$ to deform to a fibered one

<u>For today:</u> ICY/IPS fibered in AV (necessary for IPS)

Why can Gross only deduce birational boundedness?

<u>General principle:</u> ① show birational boundedness (e.g., X bir to a subvariety of $\mathbb{P}^{2\dim X+1}$ with given Hilbert polynomial)

② single out the birational model where a certain divisor is ample (e.g., $\pm K_X$)

⚠ Can have $X \underset{bir}{\simeq} X'$, $X \not\cong X'$, both ICY/IPS

<u>Solution:</u> "control" how many iso classes of ICY/IPS X' are birational to a given X

Thm. (FHS): • Develop a framework in families to achieve it (based on Kawamata–Morrison conj & HMP)
• Prove it for elliptic fibered
• Elliptic ICY 3-folds are bounded

Now, I would like to focus on how to achieve bir bdd & highlight difficulties of higher dimensions

Idea: • Given a class $\{X_i\}_{i\in I}$ s.t. each $X_i \to Y_i$, try to achieve

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[dl] \\ T & \end{tikzcd}$$

s.t., $\forall X_i$, $\exists t \in T$, $(X_i \to Y_i) \overset{\text{bir}}{\cong} (\mathcal{X}_t \to \mathcal{Y}_t)$

• If $X \to Y$ is an abelian fibration X_η is a torsor over $\mathrm{Alb}(X_\eta)$, which is AV over $\mathbb{C}(Y) = \eta$.
First understand $X^{\mathrm{Alb}} \to Y$ where $X^{\mathrm{Alb}}_\eta = \mathrm{Alb}(X_\eta)$) via moduli AV

Sketch: (1) birationally bound the bases $\{Y_i\}_{i\in I}$
• $\dim X = 3$: classification (what Gross used)
• $\dim X \geq 4$: Y_i is r.c. by irreducibility, use boundedness results of Birkar following BAB ideas

(2) the canonical bundle formula gives $f: X_i \to Y_i$

$$K_{X_i} \sim f^*(K_{Y_i} + B_{Y_i} + M_{Y_i})$$

where $B_i \geq 0$ measures singularities of fibers and M_{Y_i} measures variation of fibers

Actually bound there, so bound

$$\underbrace{Y_i \overset{\Phi}{\dashrightarrow} A_{g,\underline{d}}}_{\text{make sense by bounding graph}} \quad \text{same polarization type}$$

$\uparrow$ $\dim X - \dim Y$

coarse moduli space of polarized AV

Issues: (a) if $g \geq 2$, possibly countably many $\underline{d}$

(b) $X_i^{Alb} \to Y_i$ is retrieved (birationally) by $Y_i \dashrightarrow \mathcal{A}_{g,\underline{d}}$, not $Y_i \dashrightarrow A_{g,\underline{d}}$ (not fine moduli)

(if $(\dim X, \dim Y) \in \{(3,2),(2,1)\}$ can bypass by Weierstrass eq.)

So, we need to address issues that do not occur in Gross' work

(b) understanding B allows to control in how many ways we can lift

$$\begin{tikzcd} & \mathcal{A}_{g,\underline{d}} \arrow[d] \\ Y_i \supseteq_{op} U_i \arrow[ur, "\exists"] \arrow[r, "\Phi"'] & A_{g,\underline{d}} \end{tikzcd}$$

(a) if Z is an AV, $Z^4 \oplus (Z^*)^4$ is PPAV (Zarhin trick)

so $$\begin{tikzcd} Y_i \arrow[r, dashed, "\Phi"] \arrow[rr, dashed, bend right] & A_{g,\underline{d}} \arrow[r, "Zar"] & A_{8g,\underline{1}} \end{tikzcd}$$ bound there in a unique target

Q: can we undo the Zarhin trick in finitely many ways?

A: • in general no
• if $h^1(X,\mathcal{O}_X)=0$, or X IPS, controlling $\mathrm{Zar}\circ\Phi$ determines $\underline{d}$ and there are only finitely many possible $\underline{d}$'s
(Deligne's thm of fixed part, use lattice polarizations)

Finiteness of $\underline{d}$ + lift to $\mathcal{A}_{g,\underline{d}} \Rightarrow X^{Alb} \to Y$ bir bdd

③ Bound possible ICY/IPS twists of $X^{Alb} \to Y$

Why twists?

$$\begin{tikzcd} X \arrow[d, "f"'] & X^{Alb} \arrow[dl, "f^{Alb}"] \\ Y & \end{tikzcd}$$

if $P \in Y$, f smooth over P, X, X^{Alb} are isomorphic as complex manifolds in a nbh of X_P

on the overlaps, the gluing of the "cylinders" differ by translations of AV

$\Rightarrow$ the twists are parametrized by the H^1 of a suitable group scheme, called Tate-Shafarevich group

In EFGMS, we study this group, control its size and show varies constructibly in families in order to perform twists of the bounding family for $X^{Alb} \to Y$

Thm. (EFGMS): Fix a dimension n.

- ICY fibered in AV are bir. bdd ($\Rightarrow$ ICY fibered 3-folds bdd)
- IPS admitting a deformation that is fibered are analytically bounded